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T_1 SPACES OVER TOPOLOGICAL SITES*

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It is well known that such spaces as the (Dedekind) reals R, the Baire space B, and the Cantor space C are represented in spatial topoi as sheaves of (germs of) continuous maps to the appropriate formal space. Special cases of such representations go back to Lawvere and Tierney [3]; a more general understanding is due to Joyal and Reyes. Joyal's theory of formal spaces is expounded in Fourman and Grayson [1].

The purpose of this note is to extend such representations to a wider class of Grothendieck topoi. We need some definitions and a lemma.

Definition. A locale X is T_1 if, whenever there are morphisms $f, g: Y \to X$ from another locale Y, with $f^* \leq g^*$ pointwise, then $f = g$.

(Some take exception to this terminology since, as Johnstone has remarked, some classical T_2 spaces are not T_1 locales in this sense.)

Definition. A retraction of locales $Y \underset{i}{\overset{r}{\leftrightarrows}} Z$ is an adjoint retract if $r_* = i^*$ so that $i^* r^* \leq \mathrm{id}$.

Lemma. *If Y is an adjoint retract of Z and X is T_1 then $\mathscr{C}[r, X]$ is a natural isomorphism $\mathscr{C}[Y, X] \cong \mathscr{C}[Z, X]$.*

Proof. Clearly $\mathscr{C}[i, X]$ is a one-sided inverse. As X is T_1, from $f^* i^* r^* \leq f^*$ we deduce $rif = f$. Thus we have a two-sided inverse.

If $\mathbb{E}$ is a Grothendieck topos, $U \in \mathrm{Ob}(\mathbb{E})$, we write U^* for the locale whose opens are $\Omega(U)$, the subobjects of U.

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Definition. If X is a locale and $\mathbb{E}$ a Grothendieck topos, the sheaf $X_{\mathbb{E}}$ of *models of X in* $\mathbb{E}$ is given by $X_{\mathbb{E}}(U) \cong \mathscr{C}[U^*, X]$. [This is the sheaf of points in $\mathbb{E}$ of $\Delta(\mathscr{O}(X))$.]

Examples. The internal spaces R, B, C and $\mathscr{P}(N)$ arise as the sheaves of models of the corresponding external formal spaces. Of these, all but $\mathscr{P}(N)$ are T_1.

Definition. A *topological site* is a category of locales and continuous maps including enough open inclusions (a basis for each locale) equipped with a topology such that the natural map

$$\mathscr{O}(U) \to \Omega(U),$$

taking an open to the closure of the crible of morphisms which factor through it, has a $\vee$-preserving right adjoint. [Qua locales, this makes U an adjoint retract of the locale U^* whose opens are $\Omega(U)$.]

Examples. Any category of locales with enough open inclusions, equipped with the open cover topology or the topology of covering by open maps, is topological.

Theorem. *If X is a T_1 locale and $\mathbb{E}$ the topos of sheaves over a topological site then*

$$X_{\mathbb{E}}(U) \cong \mathscr{C}(U, X).$$

Proof. $X_{\mathbb{E}}(U) \cong \mathscr{C}(U^*, X) \cong \mathscr{C}(U, X)$.

References

[1] M.P. Fourman and R. Grayson, Formal spaces, in: A.S. Troelstra, Ed., Proc. L.E.J. Brouwer Centenary (North-Holland, Amsterdam, 1982).

[2] J.W. Gray, The meeting of the midwest category theory seminar in Zurich, in: J.W. Gray, Ed., Reports of the Midwest Category Seminar V, Lecture Notes in Math. No. 195 (Springer, Berlin, 1971) 248–255.

[3] F.W. Lawvere and M. Tierney, Lectures on elementary topoi (Zurich, August 1970) (Reported in Gray [2]).